Chapter 4

Dimensionless expressions

Dimensionless numbers occur in several contexts. Without the need for dynamical equations, one can draw a list (real or tentative) of physically relevant parameters, and use the Vaschy-Buckingham theorem to construct a shorter dimensionless list. Dimensionless expressions are the required tool to compare data from different experiments (e.g. parachute data in water), leading to the recommendation that all data should be plotted in dimensionless form. This is generally covered at the undergraduate level, and a few points of interpretation are added here. The same dimensionless expressions are obtained from dynamical equations, when available: the meaningful dimensionless numbers are ratios of terms in various equations, measuring their relative importance. This can be used to approximate the equations rationally, by dropping small (dynamically inactive) terms. One notable exception is when the small parameter is the coefficient of the highest-order derivative in the equation...

4.1 Dimensional analysis

This material is assumed known from undergraduate courses: fill in any gaps (and practice) by consulting undergraduate textbooks from the library reserve. We will review the procedure (same for all problems) with one familiar example as illustration. Occasional features not shown in the example will be mentioned for reference.

1. The list of parameters:
(a) This step determines the eventual solution, and several attempts may be necessary to identify the list that makes sense of the data. The list should be sufficiently complete to account for the physics of the flow, but not to the point of introducing unnecessary complications: trial-and-error, from the simpler to the more elaborate, is sensible. The list is a reflection of individual insight and of the profession’s expertise.

(b) Example: We will work with fully developed flow in a circular pipe. The list of parameters includes: flow parameters (pressure drop per unit length $\frac{dp}{dx}$, average speed $V$), pipe configuration (diameter $D$ or cross-sectional area $A$, not independent of course), and material properties (fluid density $\rho$ and viscosity $\mu$).

\[ \frac{\delta p}{L} \quad V \quad D \quad \rho \quad \mu \]

not included: surface roughness; etc.

(c) If some relations are known (e.g. between velocity, cross-sectional area and volume flow rate), the corresponding parameters are not independent, and one of them can be eliminated from the list for each such relation. Also, in this problem, we start from the pressure drop per unit length of pipe, rather than pressure drop and overall length as separate variables: the assumed proportionality between them is a valuable insight (try solving without it and note the differences!)

2. Primary dimensions:

(a) ‘Dimensions’ are more general than ‘units’; e.g. meter, foot and mile and micron are units relevant to the dimension of length. In general, there are 3 dimensions (M, L, T for mass, length and time respectively - other combinations can be used) for mechanical problems (some dimensions may be irrelevant in some problems, e.g. mass in simple pendulum), and then one each for thermal problems ($\theta$ for temperature), chemical, electromagnetic and radiation problems. The maximum is 7, use only as many as needed, 3 or 4 are most common in mechanical engineering problems.

(b) Example:
The dimensions of each of the problem parameters are listed. When not obvious, use a simple relation (e.g. pressure is force
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per unit area, force is mass times acceleration, etc.)

\[
\begin{align*}
\frac{\delta p}{L} & \quad V \quad D \quad \rho \quad \mu \\
ML^{-2}T^{-2} & \quad LT^{-1} \quad L \quad ML^{-3} \quad ML^{-1}T^{-1}
\end{align*}
\]

3. Select the scaling parameters:

(a) This is the next critical step. One must select as many scaling parameters (those that serve as dimensional yardsticks for all others) as there are independent dimensions. The scaling parameters must contain all dimensions in such a way that one cannot make a dimensionless expression between them; options include the selection of the simplest expressions, and/or the exclusion of the parameters you wish to solve for (see interpretation, below), called ‘control parameters’

(b) Example:
In this instance, we want to know about pressure drop, so we set it aside if possible; speed and diameter are simple and would be selected; then we need mass as a dimension, and density is simpler so we select it. 3 dimensions, 3 scaling parameters:

\[
\begin{align*}
\frac{\delta p}{L} & \quad V \quad D \quad \rho \quad \mu \\
ML^{-2}T^{-2} & \quad LT^{-1} \quad L \quad ML^{-3} \quad ML^{-1}T^{-1}
\end{align*}
\]

Pressure drop and viscosity are our control parameters in this case.

(c) Occasionally, some groupings of dimensions (e.g. \(LT^{-1}\)) may have to serve as a single dimension: go back one step and start again.

4. Non-dimensionalize each of the control parameters

(a) Then, repeat the following procedure for each of the control parameters in turn: take a parameter, multiply it by powers of the scaling parameters, and adjust the exponents to make the expression dimensionless.

(b) Example:

\[
\Pi_1 = \frac{\delta p}{L} V^a D^b \rho^c
\]

\[
1 = ML^{-2}T^{-2} L^a T^{-a} L^b M^c L^{-3c}
\]

\[
1 = M^{1+c} T^{-2-a} L^{-2+a+b-3c}
\]
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\[ c = -1 \quad a = -2 \quad b = 1 \]
\[ \Pi_1 = \frac{\delta p D}{L \rho V^2} \]

Similarly we find
\[ \Pi_2 = \frac{\mu}{\rho V D} \quad (4.1) \]

Always check these results! Always double-check the initial dimensions! Expect for standard expressions (e.g. Re) to come out.

(c) The procedure involves a system of linear equations for the exponents of each scaling parameter. The proof of the Vaschy-Buckingham theorem (see undergrad text) is based on the rank of the corresponding matrices.

5. The result

(a) The idea is that there is a relation between the parameters in the initial list; since any relation, reflecting fundamental laws and phenomenology too complicated to unravel analytically, must be dimensionally correct, it can be rearranged as a relation between dimensionless parameters. Since there are fewer of these, the relation is much simpler.

(b) Example:
In our case, we have reduced the problem of friction in pipe flows to the relation
\[ \Pi_1 = F(\Pi_2) \]
\[ \frac{\delta p}{L} = \frac{\rho V^2}{D} \psi(Re_D) \]
\[ \frac{\delta p}{\rho g} = \frac{V^2 L}{2g D} f(Re_D) \]

where \( F, \psi \) and \( f \) are as-yet unspecified functions. We recognize the Reynolds number, and we recover the familiar definition of the Darcy friction factor in the Moody diagram. Note that the relation involves an unknown function, not necessarily a proportionality.

(c) It is customary to express the pressure drop in terms of the height of a column of fluid: the parameter \( g \) is related to this scenario
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of pressure measurement, *not* to the friction in the pipe, hence it
should not be included in the initial list of parameters and serves
only for the presentation of the result.

6. Rearrangements and interpretation

(a) The procedure explained above gives a relation between the di-
mensionless control parameters. Given a result (e.g. reduced ex-
perimental data) in the form \( \Pi_1 = F(\Pi_2, \Pi_3, ...) \), you can deter-
mine their meaning by noting that the scaling parameters appear
in more than one of the \( \Pi \)s, whereas the control parameters ap-
pear in only one each.

However, you may wish to present the results differently: say you
want to know how the pressure drop depends on flow speed (which
looks similar to the original presentation by Hagen). This could
be obtained by going back to the selection of scaling parameters,
and taking viscosity instead of velocity for scaling purposes (do it
for practice: note that the algebra is a little more complicated); a
better alternative is to rearrange our previous result by combining
\( \Pi_1 \) and \( \Pi_2 \) to change scaling parameters.

(b) Example:
Between \( \Pi_1 \) and \( \Pi_2 \) as above, we want to eliminate \( V \) as a scal-
ing paramater, i.e. \( V \) should appear in only one dimensionless
product. This is done easily by combining \( \Pi_1 \) with \( \Pi_2 \)

\[
\Pi_3 = \Pi_1/\Pi_2^2 = \frac{\delta p}{L} \frac{D^3}{\rho \nu^2}
\]  

(The aspect ratio \( D/L \) is included, although starting with the
pressure drop per unit length of pipe does not bring it out as a
parameter.) The corresponding (Hagen) plot shows (dimension-
less) pressure drop as a function of flow speed (Reynolds number),
whereas the Moody plot shows pressure drop as a function of in-
verse viscosity (Reynolds number) (Fig. 4.1). Think about it.
Same data, same result, different presentation, you must read it
correctly.

Standard dimensionless numbers are tabulated in a number of undergrad-
uate texts. The student should be familiar enough with them to recognize
them when they arise.


Figure 4.1: Comparison of Moody and Hagen diagrams for friction in developed pipe flows

4.2 Non-dimensionalization of equations

This material also appears in many undergraduate texts, which should be consulted by the students. Only a few points are added here and in later chapters.

Consider the Navier-Stokes equations

$$\partial_t u_i + u_j \partial_j u_i = -\frac{1}{\rho} \partial_i p + \nu \partial^2_{jj} u_i. \quad (4.3)$$

Although one would expect 3 independent dimensions (as for most mechanical problems), we factored out density, so M is no longer a relevant dimension. So, as for dimensional analysis, we should only use 2 scaling quantities; the usual choice is to select a velocity U and length L as scaling quantities. The main point here is that the introduction of an additional pressure or time scale is unnecessary and possibly inconsistent. Asterisks will denote dimensionless quantities, for example:

$$u_i = U u_i^* \quad x_j = L x_j^*$$
Figure 4.2: About dimensional analysis
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Simple substitution gives

\[ U \partial_t u_i^* + U^2/L u_j^* \partial_j^* u_i^* = \frac{1}{\rho L} \partial_t^* p + \nu U/L^2 \partial_{jj}^* u_i^* \]  \hspace{1cm} (4.4)

It is customary (with the notable exception of Stokes flows: see Chapter 6) to adopt the nonlinear term as the yardstick and to compare all others to it by dividing throughout by \( U^2/L \).

\[ \frac{L}{u} \partial_t u_i^* + u_j^* \partial_j^* u_i^* = -\frac{1}{\rho U^2} \partial_t^* p + \frac{\nu}{UL} \partial_{jj}^* u_i^* \]  \hspace{1cm} (4.5)

We now see why it was unnecessary to select time and pressure scales: they fall out of the equations. For time, \( L/U \) is the obvious time scale; similarly, \( \rho U^2 \) is the measure of pressure scale consistent with the choice of reference term. In problems where independent time and/or pressure scales are imposed (rather than generated by the dynamics), the above presentation needs to be modified accordingly. Denoting \( t^* = tU/L \) and \( p^* = p/\rho U^2 \), we have the dimensionless Navier-Stokes equations

\[ \partial_t^* u_i^* + u_j^* \partial_j^* u_i^* = -\partial_t^* p^* + \frac{1}{Re_L} \partial_{jj}^* u_i^* \]  \hspace{1cm} (4.6)

with only the Reynolds number \( Re_L = U L/\nu \) as a parameter. When the boundary conditions of a given problem are also non-dimensionalized, this shows that all problems with same geometry and forcing (boundary conditions) and same Reynolds number will obey the same dimensionless equations, and therefore have the same solution. This is as useful for numerical simulation as for experimental comparison.

4.3 Dimensionless Equations and Scaling Analysis

All terms in an equation must have the same dimensions. Without this, changes in units would change the ratio between terms, which is physically impossible. In undergraduate classes (usually in fluid mechanics and in heat transfer), factoring out the dimensions is performed so as to obtain useful dimensionless numbers.
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In scaling analysis, we go one step further. We attempt to use the physically different length scales, say, for each term, so the scales can be vastly different in different directions. Similarly, the velocity components may scale differently, and this should be reflected when factoring out their magnitude. This leads to a proliferation of dimensionless numbers: for example, in a boundary layer, one can define a Reynolds number based on distance from the leading edge, on boundary layer thickness, on momentum thickness, etc. In complex flows such as atmospheric motion, the correct insight may depend on the scaling choices.

Consider a term such as $\partial_y u$. In dimensional analysis, one selects a length scale $L$ and a velocity scale $U$, and factor out the dimensions

$$\partial_y u = \frac{U}{L} \partial_y u^*$$  \hspace{1cm} (4.7)

Eventually dividing by $\frac{U}{L}$ yields the dimensionless form of the corresponding equation. In scaling analysis, the perspective is to get a finite difference estimate for the partial derivative: $U$ and $L$ would be such that

$$\partial_y u \sim \frac{U}{L}$$  \hspace{1cm} (4.8)

With dimensions as an underlying requirement, the emphasis shifts to having the correct order of magnitude, with the dimensionless partial derivative being replaced by a number roughly comparable to 1 (it could be 3 or 0.2, but not 100). Thus, instead of obtaining an exact dimensionless partial-differential equation, one gets an approximate algebraic equation.

Take the Navier-Stokes equations

$$\partial_t u_i + u_j \partial_j u_i = -\frac{1}{\rho} \partial_i p + \nu \partial^2_{jj} u_i$$  \hspace{1cm} (4.9)

and assume for the time being that the same scaling $U$ and $L$ applies in all directions. Then the nonlinear term scales as

$$u_j \partial_j u_i \sim \frac{U^2}{L}$$  \hspace{1cm} (4.10)

and the viscous term as

$$\nu \partial^2_{jj} u_i \sim \nu \frac{U}{L^2}$$  \hspace{1cm} (4.11)
Let us now assume that the flow is such that the convective (nonlinear) term is important; we take \( L \) as the yardstick against which the other terms will be evaluated. Then, we have

\[
\frac{L}{U^2} \partial_t u_i + 1 \sim -\frac{L}{\rho U^2} \partial_i p + \frac{v}{UL} \tag{4.12}
\]

Note that the equality is replaced by an order-of-magnitude estimate. Now, unless there is a separate mechanism (forcing) to impose a distinct time-scale, the time-derivative term can at most be of order 1 (if larger, it makes the convective term negligible, contrary to our assumptions!). Therefore

\[
\frac{L}{U^2} \partial_t u_i < 1 \tag{4.13}
\]

which is consistent with a time scale of order \( L/U \). Similarly for the pressure term, assuming no distinct length scales gives

\[
p \sim \rho U^2 \tag{4.14}
\]

as the order of magnitude for pressure variations over distances of order \( L \).

In this scenario, the orders of magnitude are: less than or comparable to 1 for evolution occurring over times not shorter than \( L/U \); order 1 for the convective term; order 1 for pressure; and order \( \nu/UL \) for the viscous term.

Watch out, use the correct scales!

### 4.4 Rational approximations

Thus, in scaling analysis, the dimensionless parameters are estimates of the relative orders of magnitude of the various terms in an equation, provided the correct parameters have been used. We saw, above, that this can provide orders of magnitude for some parameters so the corresponding terms are comparable to the leading term: time scale and pressure in the previous example. But in other instances, all terms are known (e.g. the viscous term), and their order of magnitude is of critical interest.

In the NS example, the convective term was taken, somewhat arbitrarily, as reference, and is of order 1. If the Reynolds number is large under the correct scaling, this indicates that the viscous term is relatively small: this is a rational basis for dropping the viscous term (even though large Reynolds
Figure 4.3: Comparing dimensional and scaling analysis.
number is not the same thing as inviscid: more on this below). Then the
equations of motion can be simplified as
\[
\partial_t u_i + u_j \partial_j u_i = -\frac{1}{\rho} \partial_i p
\]
and we have basis for using Euler’s equation, instead of Navier-Stokes. We
will study this case in Ch. 5
Conversely, the scaling might indicate that the Reynolds number is very
small, in which case the viscous term is much larger than the convective term.
The standard assumption of using the convective derivative as references is
invalidated: we need to go back to the first step, and instead of Eq. (4.12),
we get
\[
\frac{L^2}{U \nu} \partial_t u_i + \frac{UL}{\nu} \sim -\frac{L^2}{\mu U} \partial_i p + 1
\]
The most obvious is that the convective term can be dropped, yielding Stokes’
equation
\[
\partial_t u_i = -\frac{1}{\rho} \partial_i p + \nu \partial_j^2 u_i
\]
to be studied in Ch. 6. A second consequence, less obvious at first, is that
the natural time scales and pressure scales are different from the large-Re
case. Think about this! The pressure now scales with \( \mu U / L \) (i.e. with a
viscous stress), and the time scale is given by \( L^2 / \nu \).

The same rationale provides useful simplifications in other situations.
Unsteady forcing may be treated as quasi-static or as an instantaneous im-
pulse, depending on how the time constants match up; the earth’s rotation
may dominate the dynamics (large scale atmospheric motion) or be neglected
(bathtub vortex); etc. We will touch on these topics in later chapters. But
one case needs special consideration, and is so important that a separate
subsection seems indicated for emphasis.

### 4.4.1 Large Re

What is different about the viscous term is that it contains the highest deriva-
tive in the equation. As a general property of such equations, externally
imposed scales (\( L \) above) are not sufficient to describe the physics, since the
viscous term carries the ability to enforce the no-slip condition.

This quandary is at the core of D’Alembert’s paradox (see Ch. 5), which
divided the fluid mechanics community for the better part of the nineteenth
Advanced topics and ideas for further reading

Many examples of scaling analysis appear in turbulence theory and in convective heat transfer. After dimensional analysis, where the dynamical equations are not even used, and control volume analysis, where we integrate over many details, scaling analysis is arguably the simplest way to learn from partial differential equations.

In the limit of very large Re, energy dissipation does not behave as simply as the formula (Ch. 3) suggests: as the flow becomes turbulent, the scaling of the rate-of-strain no longer follows the externally imposed length or velocity scales, but instead follows the scaling of the turbulence. Thus, the dissipation rate becomes independent of viscosity! See your turbulence course for more on this surprising result, which shows again that $Re \to \infty$ is not a simple limit. Another instance is the boundary layer (see Ch. 8).

Problems

Some of these problems have been adapted from Tritton’s, from Fox and McDonald’s and from Mumson and Okiishi’s books.

1. Consider the vortex shedding behind a cylinder. The shedding frequency $f$ is assumed to be a function of diameter $d$, flow speed $V$, and fluid properties $\rho$ and $\mu$. Determine the form of the relation between dimensionless frequency and velocity.
Figure 4.4: The large-Re conundrum
2. The size $d$ of droplets produced by a liquid spray nozzle is thought to depend on the nozzle diameter $D$, jet velocity $U$, and the properties of the liquid $\rho$ (density), $\mu$ (dynamic viscosity) and $\sigma$ (surface tension, which is an energy per unit area). Construct the dimensionless products to show the dependence of drop size on surface tension and speed; modify the result to express the dependence of drop size on viscosity and surface tension.

3. The lift force on a Frisbee is thought to depend on its rotation speed, translation speed and diameter as well as air density and viscosity. Determine the dimensionless parameters in the relation showing lift as a function of the two speeds. Then modify the relation to express the dependence of lift on diameter.

4. The shaft power input $P$ into a pump depends on the volume flow rate $Q$, the pressure rise $dp$, the rotational speed $N$, the fluid density $\rho$, the impeller diameter $D$, and the fluid viscosity $\mu$. Express the dimensionless dependence of power on flow rate, pressure rise and other applicable parameters; discuss alternatives. What happens to your result if you learn eventually that $P = Q dp$?

5. The power $P$ required to drive a fan depends on the fluid density $\rho$, on the fan diameter $D$ and angular speed $\omega$, and on the volume flow rate $Q$. Derive the dimensionless relation showing how power depends on diameter. Reformulate the result to show how power depends on flow rate.

6. Incompressible steady flow in a magnetic field combines the equations of motion (with addition of the Lorentz force) \( \nabla \cdot \mathbf{u} = 0, \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho \mu} (\nabla \times \mathbf{B}) \times \mathbf{B} + \nu \nabla^2 \mathbf{u} \) with the equations for magnetic induction \( \nabla \cdot \mathbf{B} = 0, \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} + \frac{1}{\sigma \mu} \nabla^2 \mathbf{B} \). (here, $\mu$ is magnetic permeability and $\sigma$ is electrical conductivity). What are the similarity parameters? Discuss their physical meaning. (Problem 21 p.474 from Tritton.)

7. Dimensional analysis for pulsatile flow in a pipe: discuss the additional parameters and carry out the analysis.

8. Dimensional analysis of laminar flow in a helical pipe: discuss the additional parameters and find the dimensionless products.
9. Discuss the use of the sphere drag data in relation to improved fuel economy at lower highway speeds.

10. Scaling analysis of Bernoulli’s equation, taking $U$ and $L$ as scaling parameters. Under what conditions can we ignore gravity? Read up on the corresponding dimensionless number and give a half-page summary.

11. Use scaling analysis to find the time scale of relaxation for 1-D thermal conduction $\partial_t T = \alpha \partial^2_T T$